An Approximation for the Characteristic Impedance of Shielded-Slab Line

HENRY J. RIBLET, FELLOW, IEEE

***Abstract*—It is shown how the parametric expressions for t/b and w/b of shielded-slab line, given by means of elliptic integrals in terms of two independent real parameters a and k, can be inverted. First, $q'=\exp(-\pi K/K')$ is expressed as an odd power series in $\exp(-\pi w/(b-t))$ whose coefficients are irrational functions of t/b. Here, w is the width of the slab, t is its thickness, and b is the spacing between the infinite parallel plates. Then k is expressed in terms of q' by a well-known formula, and an expression is derived which gives a in terms of q' and $w/(b-t)$.**

When the series expressions for a and k are substituted in the formulas which yield the total capacitance C_0, of the shielded-slab line, it is found that the fringing capacitance $C_{f_0}=C_0/4-w/(b-t)$, is given by an even power series in $\exp(-\pi w/(b-t))$, at least to the eighth power. These coefficients, which are irrational functions of t/b, are given explicitly. Finally, comparisons are made with exact values of Z_0.

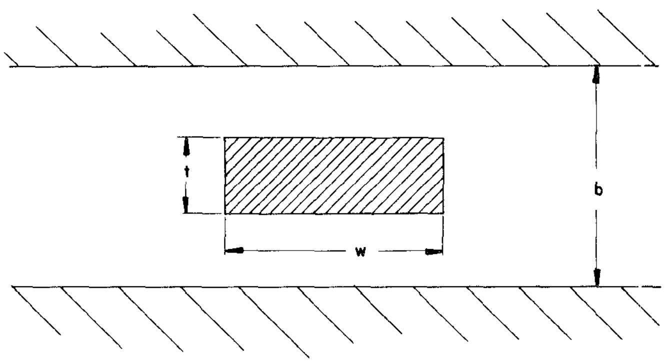

Fig. 1. Slab line geometry.

I. Introduction

In order to determine the capacitance of shielded-slab line, Bates [1] has mapped the upper half-plane into the interior of one quadrant of the infinite polygon of Fig. 1. This mapping determines the normalized dimensions of the figure, t/b and w/b, in terms of two independent parameters k and a by means of

$$t/b=1-\frac{a}{K}-\frac{2K'}{\pi}Z(a) \tag{1}$$

$$w/b=\frac{2K}{\pi}Z(a). \tag{2}$$

Here the Jacobi elliptic function $Z(a)$, and the complete elliptic integrals K and K' depend on the modulus k, for which $0<k<1$ may be assumed without restriction; and a is a real parameter for which $0<a<K$.

Following Bates, the capacitance of the structure depends on a second elliptic-function modulus k_0, which is given most simply for our purposes by

$$k_0'=\frac{\mathrm{cn}\,(a,k)}{\mathrm{dn}\,(a,k)}. \tag{3}$$

Thus k_0 also depends on the two parameters k and a. Finally the capacitance of the structure C_0 is given by

$$C_0=4K_0'/K_0 \tag{4}$$

where K_0 and K_0' are the complete elliptic integrals of the first kind of modulus k_0.

The determination of the capacitance of shielded-slab line of given dimensions using these formulas involves not only the general inconvenience of evaluating elliptic functions but also the real difficulty of solving (1) and (2) for k and a after t/b and w/b are given. It is the object of this note to show first how (1) and (2) can be inverted so that k and a may be expressed in terms of t/b and w/b, and then to substitute these values in (3) and (4). In its final form, $C_0-4w/(b-t)$ is approximated by five terms in an even power series in $\exp(-\pi w/(b-t))$ whose coefficients are irrational functions of t/b.

II. The Procedure

Eliminating $Z(a)$ between (1) and (2):

$$\frac{w}{b-t}=\frac{K}{K'}-\frac{a}{K'(1-t/b)} \tag{5}$$

and then introducing $q'=\exp(-\pi K/K')$ from the theory of elliptic functions [2, p. 225]; after putting $\alpha=1-t/b$, $w_0=w/(b-t)$, and $p=\exp(-\pi a/K')$, this becomes

$$q'=p^{1/\alpha}e^{-\pi w_0}. \tag{6}$$

It is convenient to consider for the moment large values of w_0.[1] For fixed values of t/b this will happen for large values of K. This implies in turn that $k\to 1$. The relationship between a and k that is satisfied for fixed t/b as $k\to 1$ is given by (1). That is

$$\frac{2K'}{\pi}\left(Z(a)+\frac{\pi a}{2KK'}\right)=\alpha. \tag{7}$$

Now if the substitution $a=K-a'+jK'$ is made in (7), it is found that

$$j\frac{2K'}{\pi}Z(ja',k')=1-\alpha. \tag{8}$$

[1]As will be seen later, this can mean values as small as 0.1.

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The author is with Microwave Development Laboratories, Inc., Needham Heights, MA 02194.

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This follows since

$$Z(K-a'+jK')=\frac{\operatorname{sn} a' \operatorname{dn} a'}{\operatorname{cn} a'}-Z(a')-j\frac{\pi}{2K} \tag{9}$$

from [2, p. 294, eq. 9] and [2, p. 350], while from [2, p. 293, eq. 4]

$$Z(a)+\frac{\pi a}{2KK'}=\frac{\operatorname{sn} a \operatorname{dn} a}{\operatorname{cn} a}+jZ(ja,k'). \tag{10}$$

Now from [2, p. 295, eq. 2]

$$j\frac{2K'}{\pi}Z(ja',k')$$

$$=-4q'\frac{\sinh 2\nu'-2q'^3\sinh 4\nu'+3q'^8\sinh 6\nu'+\cdots}{1-2q'\cosh 2\nu'+2q'^4\cosh 4\nu'-2q'^9\sinh 6\nu'+\cdots} \tag{11}$$

where $\nu'=\exp[\pi a'/2K']$. If $p'=\exp[-\pi(K-a')/K']$, this may be written as

$$\frac{2p'-2q'^2p'^{-1}-4q'^2p'^2+4q'^6p'^{-2}+6q'^2p'^3-6q'^{12}p'^{-3}+\cdots}{1-p'-q'^2p'^{-1}+q'^2p'^2+q'^6p'^{-2}-q'^6p'^3-q'^{12}p'^{-3}+\cdots}=1-\alpha. \tag{12}$$

Neglecting all terms in q' of twelfth degree or higher, and noting that $p=-p'$, the following equation of fifth degree in p may be obtained[2]:

$$(\alpha+5)q'^6p^5+(\alpha+3)q'^2p^4+(\alpha+1)p^3+(\alpha-1)p^2+(\alpha-3)q'^2p+(\alpha-5)q'^6=0. \tag{13}$$

This is the condition that a must satisfy as $k\to 1$ for fixed values of t/b.

It has been found that this equation can be satisfied identically for all terms as high as q'^8 by the proper selection of the coefficients in the expansion,

$$p=a+bq'^2+cq'^4+dq'^6+eq'^8. \tag{14}$$

By substituting in (13), and equating each coefficient of q' to zero, five conditions are found which determine each coefficient as a rational function of those already found. For example

$$a=\frac{1-\alpha}{1+\alpha}$$

$$b=\frac{3-\alpha-(3+\alpha)a^3}{(1+\alpha)a}$$

$$\vdots$$

To determine q' as a function of $w/(b-t)$, (6) is employed. We require

$$p^{1/\alpha}=a_1+b_1q'^2+c_1q'^4+d_1q'^6+e_1q'^8. \tag{15}$$

These coefficients follow directly from those of (14) by means of the binomial theorem. Then if an expansion of the form

$$q'=a_2e^{-\pi w_0}+b_2e^{-3\pi w_0}+c_2e^{-5\pi w_0}+d_2e^{-7\pi w_0}+e_2e^{-9\pi w_0} \tag{16}$$

is assumed and substituted back into (6), the values of $a_2,\cdots,c_2$ are readily determined. For example

$$a_2=a_1$$

$$b_2=b_1a_2^2$$

$$\vdots$$

Up to this point, the results of this paper are little more than a paraphrase of a paper by the author [4], which gives similar expansions for the geometry used in determining "approximate" fringing capacitances [5]. Having found expansions for p in terms of q', and for q' in terms of $w/(b-t)$, however, this paper proceeds by finding an expansion for C_0 in terms of q' and hence, in terms of $w/(b-t)$.

Now, from (3) and [3, p. 506, eq. 6][3]

$$k_0'=\frac{1}{\sqrt{k}}\frac{1-2q'\cosh 2\nu'+2q'^4\cosh 4\nu'-2q'^9\cosh 6\nu'}{1+2q'\cosh 2\nu'+2q'^4\cosh 4\nu'+2q'^9\cosh 6\nu'}$$
$$=\frac{1}{\sqrt{k}}\frac{1-q'(p+p^{-1})+q'^4(p^2+p^{-2})-q'^9(p^3+p^{-3})}{1+q'(p+p^{-1})+q'^4(p^2+p^{-2})+q'^9(p^3+p^{-3})} \tag{17}$$

and by [2, p. 244]

$$\sqrt{k}=\frac{1-2q'+2q'^4}{1+2q'+2q'^4}. \tag{18}$$

Thus it is clear that k_0' can be expanded as a power series in q' as high as the ninth power without difficulty. Now $q_0=\exp(-\pi K_0'/K_0)=\exp(-C_0/4)$, but by [3, p. 472, eq. 37]

$$q_0=\frac{1}{2}\frac{1-\sqrt{k_0'}}{1+\sqrt{k_0'}}+\frac{1}{2^4}\left[\frac{1-\sqrt{k_0'}}{1+\sqrt{k_0'}}\right]^5+\frac{15}{2^9}\left[\frac{1-\sqrt{k'}}{1+\sqrt{k_0'}}\right]+\cdots. \tag{19}$$

Thus having a series expansion for k_0', it is perfectly straightforward to substitute in (19) and obtain an expansion for q_0 in integral powers of q'. C_0 is then found by taking the logarithm of this expansion. Then the final step consists in replacing q' by its expansion in terms of $w/(b-t)$ already found in (16).

Performing in detail all the steps of this procedure is tedious, even in the case of a purely numerical evaluation of the coefficients encountered, so the final result will be presented without a detailed derivation. It is found that

[2]It may be of future interest to point out that this equation is a great deal simpler than the one which was originally used to obtain the final results of this paper. That was derived directly from (7). Even though it was of seventh degree in p and involved odd as well as even powers of q', it appears to be equivalent to (13).

[3]In all the following equations, it is assumed that terms involving q' of tenth or higher power have been neglected.

TABLE I
EXACT AND APPROXIMATE Z_0

W/(b-t)	t/b: 0	.1	.5	.9
1	194 226	145.665	83 262	43 079
	193 504	145540	83 242	43 074
	193 928	145 620	83255	43.077
2	153 029	123 293	75 928	41 054
	153 002	123 287	75 927	41.054
	153 024	123.292	75.928	41 054

$$\pi C_0/4 \doteq \frac{\pi w}{b-t} - A - Be^{-2\pi w_0} - Ce^{-4\pi w_0} - De^{-6\pi w_0} - Ee^{-8\pi w_0} \tag{20}$$

$$A = \frac{1}{\alpha}\log\frac{1-\alpha}{1+\alpha} + \log\frac{\alpha^2}{1-\alpha^2}$$

$$B = 4\frac{(1-\alpha)^{2/\alpha-2}}{(1+\alpha)^{2/\alpha+2}}$$

$$C = 2\frac{(1-\alpha)^{4/\alpha-4}}{(1+\alpha)^{4/\alpha+4}}(6\alpha^4 - 24\alpha^2 + 31)$$

$$D = \frac{16}{3}\frac{(1-\alpha)^{6/\alpha-6}}{(1+\alpha)^{6/\alpha+6}} \cdot (3\alpha^8 - 48\alpha^6 + 234\alpha^4 - 440\alpha^2 + 276)$$

$$E = \frac{1}{3}\frac{(1-\alpha)^{8/\alpha-8}}{(1+\alpha)^{8/\alpha+8}}(84\alpha^{12} - 2160\alpha^{10} + 22974\alpha^8 - 112992\alpha^6 + 275940\alpha^4 - 325936\alpha^2 + 150193).$$

Although A–C were derived by algebraic substitution, D and E were obtained by curve fitting methods with the help of a digital computer.

The accuracy of these coefficients can be confirmed by directly expanding the capacitance, C_0', of the stripline geometry, when $t/b=0$, in powers of $\exp(-2\pi w/b)$. In fact, it is readily found that

$$C_0' \doteq 4\frac{w}{b} + \frac{4}{\pi}\left\{2\log 2 - \frac{1}{4}e^{-2\pi w/b} - \frac{13}{128}e^{-4\pi w/b} - \frac{23}{384}e^{-6\pi w/b} - \frac{2701}{65536}e^{-8\pi w/b}\right\}. \tag{21}$$

Of course, the values of the algebraic coefficients in (20) approach the coefficients in (21) as $t/b \to 0$ even though values at $t/b=0$ were not used in the curve fitting process.

The first two terms in (20) are the well-known parallel-plate capacitance C_p, and the approximate fringing capacitance C_f', as given by Cohn [6] and Smythe [7]. B, which is thought to be new, is of interest because it determines the asymptotic behavior of those curves ΔC_{f_0}, (Fig. 2, [8]) for which $s/b=\infty$.

Table I compares the exact values of the impedance of shielded-slab line with values obtained from (20) in two different ways. The upper value in each column is exact while the lowest value uses all of the terms in (20). The middle term neglects E. It is seen that for $w/(b-t) \geq 0.2$, even neglecting E, that the error is always less than 0.02 percent, and decreases rapidly as t/b increases. For $w/(b-t) \geq 0.1$, the error for $t/b=0$ is less than 0.16 percent and less than 0.03 percent for $t/b=0.5$. Since the error is an increasing function of Z_0, it can be safely stated that whenever $t/b \geq 0.1$, (20) can be used to calculate the impedance of shielded-slab line with an accuracy of 0.03 percent for all values less 150 Ω.

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